Bull. Math. Sci. (2013) 3:1–17
DOI 10.1007/s13373-012-0022-y

Rook version of colored partition algebras

A. Joseph Kennedy · G. Muniasamy

Received: 23 October 2011 / Revised: 7 March 2012 / Accepted: 12 March 2012 /
Published online: 5 April 2012

Abstract We study the rook version of the colored partition algebras $P_k(n, G)$ and $\widehat{P}_k(n, G)$ and we obtain the corresponding Schur–Weyl dualities.

Keywords Partition algebra · Centralizer algebra · Direct product · Wreath product · Symmetric group

Mathematics Subject Classification 16S20 · 16S50 · 16S99

1 Introduction

There are various deformations of semigroup algebras arising from the generalizations of the classical Schur–Weyl duality. The partition algebras $P_k(x)$ have been studied independently by Martin and Jones, as a generalization of the Temperley-Lieb algebras and the Potts model in statistical mechanics (see [7]). In 1993, Jones considered $P_k(n)$, as the centralizer algebra of the symmetric group S_n on $V^{\otimes k}$ (see [5]).

The Class (or Ramified) partition algebra $P_k(m, n)$ has been introduced in [8] also in [6] by Kennedy, and has been realized as the centralizer algebra of the wreath product $S_m \wr S_n$ acting on the tensor space $W^{\otimes k}$, where $W = \mathbb{C}^{mn}$ is the permutation module for the symmetric group S_{mn}. The G-edge colored partition algebra $\overrightarrow{P}_k(n, G)$ has been introduced in [1] by Bloss, and has been realized as the centralizer algebra

Communicated by S.K. Jain.

Research supported by Govt. of India UGC-SAP and DST Research Project No. SR/FTP/MS-007/2010.

A. Joseph Kennedy (✉) · G. Muniasamy
Department of Mathematics, Ramanujan School of Mathematics and Computer Science,
Pondicherry University, Pondicherry 605 014, India
e-mail: kennedy.pondi@gmail.com

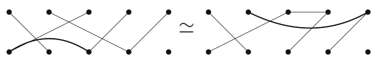
of the wreath product $G \wr S_n$ inside $S_{|G|} \wr S_n$. The G-vertex colored partition algebra $P_k(n, G)$ has been introduced in [11] and has been realized as the centralizer algebra of the subgroup $G \times S_n$ of $G \wr S_n$. The extended vertex colored partition algebras $\widehat{P}_k(n, G)$, which is the centralizer algebra of the subgroup S_n of $G \times S_n$, and the representations of these algebras have been studied in [12] and [13]. In the case $|G| = m$, the natural inclusion of groups $S_n \subseteq G \times S_n \subseteq G \wr S_n \subseteq S_m \wr S_n \subseteq S_{mn}$ induces the natural revers inclusion of the corresponding centralizer algebras as $P_k(mn) \subseteq P_k(m, n) \subseteq \overrightarrow{P}_k(n, G) \subseteq P_k(n, G) \subseteq \widehat{P}_k(n, G)$, which have been studied explicitly in [6].

The rook (or half) partition algebras have been introduced by Martin and Rollet [9], also studied by Halverson and Ram [3] and Grood [2] with different notations. We will use the notation $P_{k+\frac{1}{2}}(x)$ for the half partition algebra. The half partition algebra $P_{k+\frac{1}{2}}(x)$ is the centralizer algebra of the symmetric group S_{n-1} on $V^{\otimes k}$, where $V = \mathbb{C}^n$ is the natural representation of S_n. This rook version is used to construct the RSK correspondence for the partition algebra, see [4]. In this paper, we study the rook version of these colored partition algebras $P_k(x, G)$ and $\widehat{P}_k(n, G)$ and the corresponding Schur–Weyl dualities.

2 Preliminaries

2.1 The structure of $P_k(x)$ and the rook version

A k-partition diagram is a simple graph on two rows of k-vertices, one above the other. The connected components of such a graph partition the $2k$ vertices into l disjoint subsets with $1 \leq l \leq 2k$. We say that two k-diagrams are *equivalent* if they give rise to the same partition of the $2k$ vertices. For example, the following are equivalent 5-diagrams.



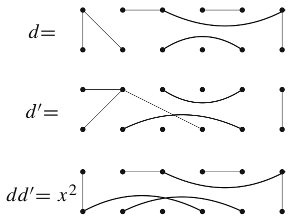
When we speak of diagrams, we are really talking about the associative equivalence classes. Number the vertices of a k-diagram $1, 2, \ldots, k$ from left to right in the top row, and $k+1, k+2, \ldots, 2k$ from left to right in the bottom row.

For every field F and $x \in F$, we can define the partition algebra $P_k(x)$ on F-span of the k-partition diagrams with the following multiplication on diagrams.

The multiplication of two k-partition diagrams d and d' is defined as follows:

- Place d on the top and d' at the bottom.
- Identify (or join) the $(k+j)^{\text{th}}$ vertex of d with the j^{th} vertex of d'. The resulting diagram now has a top row, a bottom row and a middle row of vertices.
- Let d'' be the partition diagram whose classes are obtained from the resulting diagram by using only the top and bottom row vertices in which they are connected by some path. Replace each "component" which is contained in the middle level by the variable x. (ie.) $dd' = x^{\lambda} d''$, where λ is the number of components in the middle level.

For example,



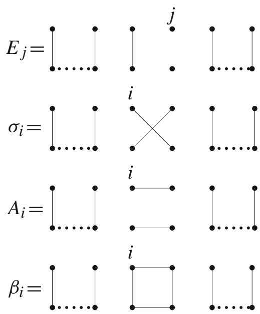
This product is associative and is independent of the graph that we choose to represent the k-partition diagram. The identity is given by the partition diagram having each vertex in the top row connected to the vertex below it in the bottom row. The dimension of $P_k(x)$ is the *Bell number* $B(2k)$ and

$$B(2k) = \sum_{l=1}^{l=2k} S(2k, l), \tag{2.1}$$

where the *Sterling number* $S(2k, l)$ is the number of equivalence relations having exactly l parts.

The span of the partition diagrams for which each component has exactly two vertices is the *Brauer algebra* $B_k(x)$. The span of the partition diagrams for which each component has exactly two vertices, one in each row is the group algebra $F[S_k]$ of the symmetric group S_k.

For $1 \leq i \leq k-1$ and $1 \leq j \leq k$, define



The elements $\{\sigma_i\}$ generate $F[S_k]$, the elements $\{\sigma_i, A_i\}$ generate the $B_k(x)$ and the elements $\{\sigma_i, \beta_i, E_j\}$ generate $P_k(x)$.

Theorem 2.1.1 [10] *For each integer $k \geq 0$, $P_k(x)$ is semisimple over $\mathbb{C}(x)$, the field of complex rational polynomials in x. The algebra $P_k(\xi)$ is semisimple over $\mathbb{C}$ whenever ξ is not an integer in the range* $[0, 2k-1]$. □

2.2 Schur–Weyl duality

Let $V = \mathbb{C}^n$ with standard basis $v_1, v_2, \ldots, v_n$ be the permutation module for the symmetric group S_n. Then $\pi(v_i) = v_{\pi(i)}$, for $\pi \in S_n$ and $1 \leq i \leq n$. For each positive integer k, the tensor product space $V^{\otimes k}$ is a module for the group S_n with a standard basis given by $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$, where $1 \leq i_j \leq n$. The action of $\pi \in S_n$ on a basis vector is given by

$$\pi(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{\pi(i_1)} \otimes v_{\pi(i_2)} \otimes \cdots \otimes v_{\pi(i_k)}. \tag{2.2}$$

For each k-partition diagram d and each integer sequence $i_1, i_2, \ldots, i_{2k}$ with $1 \leq i_s \leq n$, define

$$\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},\ldots,i_{2k}} = \begin{cases} 1 & \text{if } i_r = i_s \text{ whenever } r \sim s \text{ (i.e. } r \text{ and } s \text{ are in same class) in } d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Define the action of a partition diagram $d \in P_k(n)$ on $V^{\otimes k}$ by defining it on the standard basis by

$$d(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = \sum_{ik+1,\ldots,i_{2k}} \psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},\ldots,i_{2k}} v_{i_{k+1}} \otimes v_{i_{k+2}} \otimes \cdots \otimes v_{i_{2k}}. \tag{2.4}$$

Theorem 2.2.1 (Jones [5]) *S_n and $P_k(n)$ generate full centralizers of each other in $End(V^{\otimes k})$. In particular, for $n \geq 2k$,* (a) $P_k(n) \cong End_{S_n}(V^{\otimes k})$. (b) S_n *generates* $End_{P_k(n)}(V^{\otimes k})$.

The rook partition algebra $P_{k+\frac{1}{2}}(n)$ is the centralizer algebra of the subgroup S_{n-1} of all permutation fixing n in S_n. Hence we have $P_k(n) \subseteq P_{k+\frac{1}{2}}(n)$. The rook partition algebra $P_{k+\frac{1}{2}}(n)$ has been realized as a subalgebra of the partition algebra $P_{k+1}(n)$ as the span of all partition diagrams in which the last two vertices ($k+1$th and $2(k+1)$th) are in a same class, see for example [2,3,9].

2.3 The colored partition algebras $P_k(x, G)$

Let G be a group. We denote $[m]$ for the set $\{1, 2, \ldots, m\}$. Let $G^{2k} = \{f \mid f : [2k] \longrightarrow G\}$. We say that each $f \in G^{2k}$ is a coloring of $[2k]$ by G. We define a multiplication on G^{2k} by $ff'(p) = f(p)f'(p)$, for all $f, f' \in G^{2k}$ and $p \in [2k]$. Note that under this multiplication, G^{2k} is a group, called the *coloring group* of $[2k]$ by G.

Let $f \in G^{2k}$. We can write $f = (f_1, f_2)$, where $f_1, f_2 \in G^k$ are defined on $[k]$ by $f_1(p) = f(p), \; f_2(p) = f(k+p)$, for all $p \in [k]$. We say that f_1 and f_2 are the first and the second component of f respectively.

A (G, k)-*vertex colored partition diagram* (or simply G-diagram) is a k-partition diagram, where each vertex is labelled by an element of the group G. We can identify each G-diagram as a pair (d, f), where d is the underlying k-partition diagram and $f \in G^{2k}$ such that $f(i)$ is the label of the ith vertex.

Let (d, f) and (d', f') be two G-diagrams, where d, d' are any two k-partition diagrams and $f = (f_1, f_2), \; f' = (f'_1, f'_2) \in G^{2k}$. In [11], we defined an equivalence relation $\sim$ on G-diagrams and a multiplication on G-diagrams, which is associative and well-defined up to equivalence of such diagrams, as follows:

- $(d, f) \sim (d', f') \Leftrightarrow d \sim d'$ and $f = \overline{g} f'$ for some (unique) $\overline{g} \in \overline{G}$
  $\Leftrightarrow d \sim d'$ and $f \in \overline{G} f'$.
- $(d', f')(d, f) = \begin{cases} x^{\lambda}(d'', f'') & \text{if } f_2 = (\overline{g} f')_1 \text{ for some (unique) } \overline{g} \in \overline{G} \\ 0 & \text{otherwise,} \end{cases}$ where
  $d'd = x^{\lambda} d''$ and $f'' = (f_1, (\overline{g} f')_2)$.

When we speak of a G-diagram, we are really speaking of its equivalence class. The F-span of all $\sim$-classes of G-diagrams is denoted as $P_k(x, G)$, called the *G-vertex colored partition algebra*, which is an associative algebra with identity. For each $\sim$-class, we can choose a G-diagram (d, f) such that $f(1) = e$. Now we may consider the set $\{(d, f) \mid f(1) = e\}$ as a basis for the algebra $P_k(x, G)$. The identity in $P_k(x, G)$ is

$$\sum_{\substack{f \in G^{2k} \\ f(1)=e, f_1=f_2}} (d, f),$$

where d is the identity partition diagram. The dimension of the algebra $P_k(x, G)$ is $|G|^{2k-1} B(2k)$, if G is finite.

Let G be any finite group of order m and let W be the natural permutation module for the symmetric group S_{mn} of dimension mn. We can identify W as $\text{Span}_{\mathbb{C}}\{v_{(i,g)}/1 \leq i \leq n \text{ and } g \in G\}$. In [11], we defined a map $\phi : P_k(n, G) \longrightarrow \text{End}(W^{\otimes k})$ by defining it on a basis element (d, f) such that $f(1) = e$, as follows:

$$\begin{aligned}
\phi(d, f) &= \left(\phi(d, f)^{(i_1,h_1),(i_2,h_2),\ldots,(i_k,h_k)}_{(i_{k+1},h_{k+1}),(i_{k+2},h_{k+2}),\ldots,(i_{2k},h_{2k})}\right) \\
&= \left(\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}} \quad \delta^{h_1(f(1),f(2),\ldots,f(2k))}_{h_1,h_2,\ldots,h_{2k}}\right) \\
&= \sum_{\substack{g \in G \\ p \sim q \text{ in } d \Rightarrow i_p = i_q}} E^{(i_1,gf(1)),(i_2,gf(2)),\ldots,(i_k,gf(k))}_{(i_{k+1},gf(k+1)),(i_{k+2},gf(k+2)),\ldots,(i_{2k},gf(2k))}
\end{aligned}$$

where $\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}}$ is defined as in Eq. (2.3). We have an action of the algebra $P_k(n, G)$ on $W^{\otimes k}$ with respect to ϕ, defined by

$$\begin{aligned}
&(d, f).(v_{(i_1,h_1)} \otimes v_{(i_2,h_2)} \otimes \cdots \otimes v_{(i_k,h_k)}) = \delta^{h_1(e,f(2),f(3),\ldots,f(2k))}_{(h_1,h_2,\ldots h_{2k})} \\
&\times \sum_{1 \leq i_{k+1}, i_{k+2}, \ldots, i_{2k} \leq n} \psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}} v_{(i_{k+1},h_{k+1})} \otimes \cdots \otimes v_{(i_{2k},h_{2k})}.
\end{aligned}$$

Consider the restricted action (as explained in the introduction) of the subgroup $G \times S_n$ on W as follows: $\pi_g(i, h) = (\pi(i), gh)$. Then ϕ is a algebra homomorphism onto $\text{End}_{G \times S_n}(W^{\otimes k})$ (see, [11]).

Theorem 2.3.1 [11] *$\mathbb{C}[G \times S_n]$ and $P_k(n, G)$ generate full centralizers of each other in* $\text{End}(W^{\otimes k})$. *In particular, for $n \geq 2k$*

(a) $P_k(n, G) \cong \mathrm{End}_{G\times S_n}(W^{\otimes k})$
(b) $G \times S_n$ *generates* $\mathrm{End}_{P_k(n,G)}(W^{\otimes k})$.

The another algebra $\widehat{P}_k(n, G)$ is spanned by all G-diagrams with the following multiplication:

- $$(d', f')(d, f) = \begin{cases} x^{\lambda}(d'', f'') & \text{if } f_2 = f'_1 \\ 0 & \text{otherwise,} \end{cases}$$

where $d'd = x^{\lambda}d''$ and $f'' = (f_1,\ (\overline{g}f')_2)$.

Theorem 2.3.2 [12] $\mathbb{C}[S_n]$ *and* $\widehat{P}_k(n, G)$ *generate full centralizers of each other in* $\mathrm{End}(W^{\otimes k})$. *In particular, for* $n \geq 2k$

(a) $\widehat{P}_k(n, G) \cong \mathrm{End}_{S_n}(W^{\otimes k})$
(b) S_n *generates* $\mathrm{End}_{\widehat{P}_k(n,G)}(W^{\otimes k})$.

3 The rook version of colored partition algebras

In this section, we introduce the rook version of the colored partition algebras and study its structure.

3.1 Two bases for $End_{G\times S_{n-1}}(W^{\otimes k})$ and $End_{S_{n-1}}(W^{\otimes k})$

In this section, we give two bases for $End_{G\times S_{n-1}}(W^{\otimes k})$, where $W = \mathbb{C}^{|G|n}$ and the action of $G \times S_{n-1}$ on $W^{\otimes k}$ is defined as follows:

Let G be any finite group and let W be a vector space of dimension $|G|n$. We can identify W as $\mathrm{Span}_{\mathbb{C}}\{v_{(i,g)}/1 \leq i \leq n \text{ and } g \in G\}$. Note that when G is the group with one element, W specializes to V, the permutation representation of S_n. The action of $G \times S_{n-1}$ on W is defined as

$$\pi_g\left(v_{(i,h)}\right) = v_{(\pi(i),gh)}, \forall \pi \in S_{n-1} \quad \text{and} \quad g \in G$$

(note that S_{n-1} is the subgroup of all permutation fixing n in S_n).

Diagonally extend the action of $G \times S_{n-1}$ on W to an action of $G \times S_{n-1}$ on $W^{\otimes k}$:

$$\pi_g(v_{(i_1,g_1)} \otimes \cdots \otimes v_{(i_k,g_k)}) = v_{(\pi(i_1),gg_1)} \otimes \cdots \otimes v_{(\pi(i_k),gg_k)} \tag{3.1}$$

where $\pi \in S_{n-1}$ and $g \in G$. We will write above as $\pi_g(v_I) = v_{\pi_g(I)}$.

Let $A \in End(W^{\otimes k})$. Define $A(v_I) = \sum_J A^I_J(v_J)$, where $A^I_J \in \mathbb{C}$ is the $(I, J)^{\text{th}}$ entry of A; $I, J \in \mathbb{S}^k$, where $\mathbb{S} = [n] \times G$ and v_J is the basis element of $W^{\otimes k}$. We have

$$End_{G\times S_n}(W^{\otimes k}) \subseteq End_{G\times S_{n-1}}(W^{\otimes k}) \subseteq End_{S_{n-1}}(W^{\otimes k}).$$

The following is our analogue of Jones result.

Lemma 3.1.1 **(a)** $A \in End_{G\times S_{n-1}}(W^{\otimes k)} \Leftrightarrow A^I_J = A^{\pi(I)}_{\pi(J)}, \forall \pi_g \in G \times S_{n-1}$. **(b)** $A \in End_{S_{n-1}}(W^{\otimes k)} \Leftrightarrow A^I_J = A^{\pi(I)}_{\pi(J)}, \forall \pi \in S_{n-1}$.

Proof **(a)** We have $A \in End_{G\times S_{n-1}}(W^{\otimes k}) \Leftrightarrow \pi_g A = A\pi_g, \forall \pi_g \in G \times S_{n-1}$.

$\Leftrightarrow \pi_g A(v_I) = A\pi_g(v_I) \forall v_I$.

$\Leftrightarrow \pi_g \sum_J A^I_J(v_J) = A(v_{\pi_g(I)})$

$\Leftrightarrow \sum_J A^I_J \pi_g(v_J) = \sum_J A^{\pi_g(I)}_J(v_J)$

$\Leftrightarrow \sum_J A^I_J(v_{\pi_g(J)}) = \sum_J A^{\pi_g(I)}_{\pi_g(J)}(v_{\pi_g(J)})$, since the action of $G \times S_{n-1}$ is the permutation representation. The result (a) follows from linear independence and equating the scalars. The proof of (b) is similar to the proof of (a). □

Lemma 3.1.2

$$(\mathbf{a})\ \dim\ End_{G\times S_{n-1}}(W^{\otimes k}) = |G|^{2k-1} \sum_{l=1}^{l=n} S(2k+1, l).$$

When $n - 1 \geq 2k$,

$$\dim End_{\ G\times S_{n-1}}(W^{\otimes k}) = |G|^{2k-1}\ B(2k+1).$$

$$(\mathbf{b})\ \dim\ End_{S_{n-1}}(W^{\otimes k}) = |G|^{2k} \sum_{l=1}^{l=n} S(2k+1, l).$$

When $n - 1 \geq 2k$,

$$\dim End_{\ S_{n-1}}(W^{\otimes k}) = |G|^{2k}\ B(2k+1).$$

Proof **(a)** The lemma above tells us that A commutes with the $G \times S_{n-1}$-action on $W^{\otimes k}$ if and only if the matrix entries of A are equal on the $G \times S_{n-1}$-orbits. Thus $dim\ End_{G\times S_{n-1}}(W^{\otimes k})$ is the number of $G \times S_{n-1}$-orbits on $\mathbb{S}^{2k} = \{((i_1, g_1), (i_2, g_2) \ldots, (i_{2k}, g_{2k})) \mid 1 \leq i_r \leq n$ and $g_r \in G\}$. Fix a tuple of indices $(I, J) = ((i_1, g_1), (i_2, g_2) \ldots, (i_{2k}, g_{2k})) \in \mathbb{S}^{2k}$. This tuple determines a partition $d(I, J) = d(i_1, i_2, \ldots, i_{2k})$ of $[2k]$ (into at most n subsets) according to those that have an equal value. Let $[(I, J)]$ be the orbit of $(I, J) \in \mathbb{S}^{2k}$.

Then $(I', J') \in [(I, J)] \Leftrightarrow (I', J') = \pi(I, J)$ for some $\pi_g \in G \times S_{n-1}$

$\Leftrightarrow$ $(j_r, h_r) = \pi_g(i_r, g_r)$ for every r such that $1 \leq r \leq 2k$, where (j_r, h_r) and (i_r, g_r) are the r^{th} component of (I', J') and (I, J) respectively

$\Leftrightarrow (j_r, h_r) = (\pi(i_r), gg_r)$

$\Leftrightarrow j_r = \pi(i_r)$ and $h_r = gg_r$

$$\Leftrightarrow [j_p = j_q \text{ iff } i_p = i_q\ (1 \leq p, q \leq 2k)],\ [j_p = n \text{ iff } i_p = n\ (1 \leq p \leq 2k)] \text{ and } h_r = gg_r\ (1 \leq r \leq 2k) \tag{3.2}$$

$\Leftrightarrow d(j_1, j_2, \ldots, j_{2k}) = d(i_1, i_2, \ldots, i_{2k}),\ [j_p = n$ iff $i_p = n\ (1 \leq p \leq 2k)]$ and $h_r = gg_r \forall r, (1 \leq r \leq 2k)$.

Thus, for every $G\times S_{n-1}$-orbit $[(I, J)]$ determines a partition $d = d(i_1, i_2, \ldots, i_{2k})$ (into at most n subsets) and a class $N = \{p \in [2k] \mid i_p = n\}$ of d and a $2k$-tuple $f = (e, g_2, \ldots, g_{2k})$ (i.e a pair (d_N, f) such that $f(1) = e$) and vice versa. Hence the result (a) is proved.

Similarly, for every S_{n-1}-orbit $[(I, J)]$ determines a partition $d = d(i_1, i_2, \ldots, i_{2k})$ (into at most n subsets) and a class $N = \{p \in [2k] \mid i_p = n\}$ of d and a $2k$-tuple $f = (g_1, g_2, \ldots, g_{2k})$ (i.e a pair (d_N, f)) and vice versa. Hence the result (b) is proved. $\square$

We define for each $G \times S_{n-1}$-orbit $(d_N, f) = [(I, J)]$, a matrix $T_J^I \in End(W^{\otimes k})$ by $T_J^I = \sum_{(I',J')\in[(I,J)]} E_{J'}^{I'}$, where $E_{J'}^{I'}$ is the matrix unit, which has non-zero entry 1 in the $(I', J')^{th}$ position. In fact, $T_J^I \in End(W^{\otimes k})$, since such a matrix satisfies the condition in Lemma 3.1.1 : the matrix entries are equal on the $G \times S_{n-1}$-orbits. Using Eq. (3.2), we have

$$T^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})} = \sum E^{(j_1,gg_1),(j_2,gg_2),\ldots,(j_k,gg_k)}_{(j_{k+1},gg_{k+1}),(j_{k+2},gg_{k+2}),\ldots,(j_{2k},gg_{2k})}, \tag{3.3}$$

where the sum is over $g \in G$ and $i_p = i_q \Leftrightarrow j_p = j_q$, $(1 \le p, q \le 2k)$ [i.e $p \sim q$ in $P(i_1, i_2, \ldots, i_{2k}) \Leftrightarrow j_p = j_q$] and $i_p = n \Leftrightarrow j_p = n$.

Since each matrix T_J^I is the sum of disjoint sets of matrix units, the set $\{T_J^I/[(I, J)]$ is a $G \times S_{n-1}$ − orbit$\}$ is a linearly independent set. For $A \in End_{G\times S_{n-1}}(W^{\otimes k})$, we use the Lemma 3.1.1 to obtain: $A = \sum_{[(I,J)]} A_J^I T_J^I$. Thus the matrix T_J^I span $End_{G\times S_{n-1}}(W^{\otimes k})$, so is a basis for $End_{G\times S_{n-1}}(W^{\otimes k})$.

Definition 3.1.3 Let d and d' be partitions of $[2k]$ into subsets. We say that d' is *coarser* than d if any subset in d is contained in some subset in d'. In this case we write $d' \le d$.

We now define another basis of $End_{G\times S_{n-1}}(W^{\otimes k})$ as follows : Define $L_J^I = \sum T_{J'}^{I'}$, the sum is over $[(I', J')]$ such that $d[(I', J')] \le d[(I, J)]$. By Möbius inversion the T_J^I can be expressed in terms of the L_J^I 's so they also span $End_{G\times S_{n-1}}(W^{\otimes k})$. Using Eq. (3.3), we get

$$L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})} = \sum E^{(j_1,gg_1),(j_2,gg_2),\ldots,(j_k,gg_k)}_{(j_{k+1},gg_{k+1}),(j_{k+2},gg_{k+2}),\ldots,(j_{2k},gg_{2k})} \tag{3.4}$$

where the sum is over $g \in G$ and $i_p = i_q \Rightarrow j_p = j_q$ $(1 \le p, q \le 2k)$ and $i_p = n \Rightarrow j_p = n$.

Similarly, we use the proof of Lemma 3.1.2(b) to define for each S_{n-1}-orbit $[(I, J)]$ a matrix in $\mathrm{End}_{S_{n-1}}(W^{\otimes k})$, as follows:

$$\widetilde{L}^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})} = \sum E^{(j_1,g_1),(j_2,g_2),\ldots,(j_k,g_k)}_{(j_{k+1},g_{k+1}),(j_{k+2},g_{k+2}),\ldots,(j_{2k},g_{2k})}, \tag{3.5}$$

where the sum is over $i_p = i_q \Rightarrow j_p = j_q$ $(1 \le p, q \le 2k)$ and $i_p = n \Rightarrow j_p = n$. Note that

$$L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})} = \sum_{g\in G} \widetilde{L}^{(j_1,gg_1),(j_2,gg_2),\ldots,(j_k,gg_k)}_{(j_{k+1},gg_{k+1}),(j_{k+2},gg_{k+2}),\ldots,(j_{2k},gg_{2k})}. \tag{3.6}$$

3.2 The structure of $P_{k+\frac{1}{2}}(x, G)$ and $\widehat{P}_{k+\frac{1}{2}}(x, G)$

A $(G, k+\frac{1}{2})$-partition diagram is a (G, k)-partition diagram whose underlying partition d with a special class N of d (N may be empty) and the vertices are colored by G. Now each $(G, k+\frac{1}{2})$-diagram can be identified as a pair (d_N, f), where d is the partition of the set $[2k]$ induced by the (G, k)-diagram, N is the unique special class of d (N may be empty) and $f \in G^{2k}$ is the coloring induced by the coloring sequence of the (G, k)-diagram.

For each d_N, we can get a partition with $2k+1$ vertices by adding a $2k+1$th vertex in the special class N and vice versa. Hence the number of $(G, k+\frac{1}{2})$-diagrams is $|G|^{2k}B(2k+1)$. Thus each $(G, k+\frac{1}{2})$-diagram can be identified as a (G, k)-partition diagram with one more vertex in the right side of the diagram, which is connected with the special class N.

We define two multiplications on $(G, k+\frac{1}{2})$-diagrams, where two G-diagrams (d_N, f) and $(d'_{N'}, f')$, where d, d' are any two partitions of the set $[2k]$ and N, N' are the special classes of d, d' connected with the $2k+1$th vertex respectively and $f = (f_1, f_2),\ f' = (f'_1, f'_2) \in G^{2k}$, as follows:

(1)

$$(d_N, f)(d'_{N'}, f') = \begin{cases} x^{\lambda}(d_N d'_{N'}, (f'_1, gf_2)) & \text{if } f'_2 = gf_1 \text{ for some } g \in G \\ 0 & \text{otherwise,} \end{cases} \tag{3.7}$$

where λ is the number of middle components of $d_N d'_{N'}$ as in the partition algebra case.

(2)

$$(d_N, f) * (d'_{N'}, f') = \begin{cases} x^{\lambda}(d_N d'_{N'}, (f'_1, f_2) & \text{if } f'_2 = f_1 \\ 0 & \text{otherwise,} \end{cases} \tag{3.8}$$

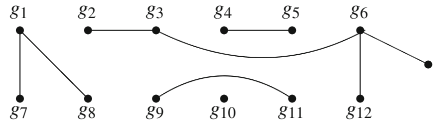
where λ is the number of middle components of $d_N d'_{N'}$.

The multiplication (1) of two G-diagrams (d_N, f) and $(d'_{N'}, f')$ defined above can be equivalently stated in other words as follows:

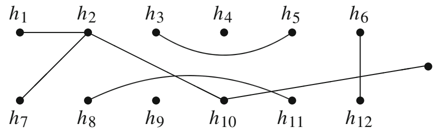
- Multiply the underlying partition diagrams d_N and $d'_{N'}$. This will give the underlying partition diagram of the G-diagram $(d_N, f)(d'_{N'}, f')$.
- If the bottom label sequence of (d_N, f) is equal to the top label sequence of $(d'_{N'}, gf')$ for some $g \in G$ then the top label sequence and the bottom label sequence of $(d_N, f)(d'_{N'}, f')$ are the top label sequence of (d_N, f) and the bottom label sequence of (d'_N, gf') respectively.
- Otherwise $(d_N, f)(d'_{N'}, f') = 0$.
- For each connected component entirely in the middle row, a factor of x (indeterminant) appears in the product.

For example for the multiplication (1), let $g_r, h_s \in G (1 \leq r, s \leq 12)$.

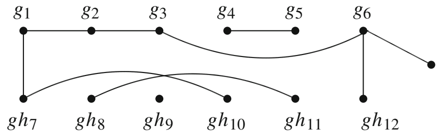
$(d,f)=$



$(d',f')=$



$(d,f)(d',f')= \quad x^2\, \delta^{(g_7,g_8,\ldots,g_{12})}_{(gh_1,gh_2,\ldots,gh_6)}$



Note that δ is the Kronecker delta, that is

$$\delta^{(g_7,g_8,\ldots,g_{12})}_{(gh_1,gh_2,\ldots,gh_6)} = \begin{cases} 1 & \text{if } (g_7, g_8, \ldots, g_{12}) = (gh_1, gh_2, \ldots, gh_6) \\ 0 & \text{if } (g_7, g_8, \ldots, g_{12}) \neq (gh_1, gh_2, \ldots, gh_6). \end{cases}$$

The multiplication (2) can be explained using diagrams in the similar way.

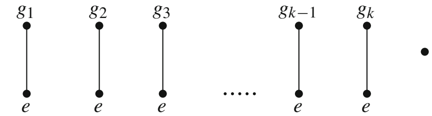
Moreover the multiplication (2) is well defined under the following equivalence relation:

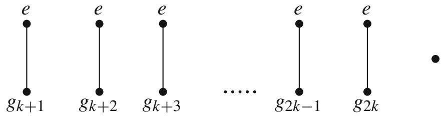
$$(d_N, f) \sim (d'_{N'}, f') \text{ iff } d_N \sim d'_{N'} \text{ and } f = gf' \text{ for some } g \in G. \tag{3.9}$$

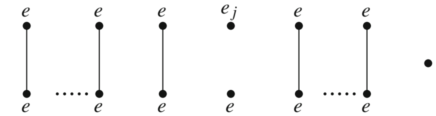
The multiplications (1) and (2) are associative on $(G, k+\frac{1}{2})$-diagrams. The $\mathbb{C}(x)$-span of all $(G, k+\frac{1}{2})$-diagrams under the above multiplication (1) with the above equivalence relation and the multiplication (2) are denoted as $P_{k+\frac{1}{2}}(x, G)$ and $\widehat{P}_{k+\frac{1}{2}}(x, G)$ respectively, which are associative algebras with identity. The identity in $P_{k+\frac{1}{2}}(x, G)$ is $\sum_{\substack{f\in G^{2k} \\ f_1=f_2,\, f_1(1)=f_2(1)=e}} (d_N, f)$ and in $\widehat{P}_{k+\frac{1}{2}}(x, G)$ is $\sum_{\substack{f\in G^{2k} \\ f_1=f_2}} (d_N, f)$, where d is the identity partition diagram and N is empty.

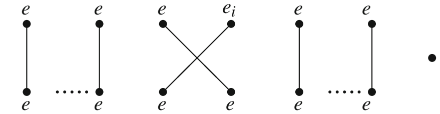
The dimension of $P_{k+\frac{1}{2}}(x, G)$ is the number of equivalence classes defined in (3.9) of $(G, k+\frac{1}{2})$-diagrams, so that if G is finite, $dim\ P_{k+\frac{1}{2}}(x, G)=|G|^{2k-1}B(2k+1)$, where $B(2k+1)$ is the Bell number of $2k+1$, the number of equivalence relations of $2k+1$ vertices. Note that $P_{k+\frac{1}{2}}(x, H)$ is a subalgebra of $P_{k+\frac{1}{2}}(x, G)$ if H is a subgroup of G. In particular, if $H = \{e\}$ then $P_{k+\frac{1}{2}}(x, H) \simeq P_{k+\frac{1}{2}}(x)$, the rook version of the partition algebra. If G is an infinite group, $P_{k+\frac{1}{2}}(x, G)$ is an infinite dimensional associative algebra. When $x = \xi \in \mathbb{C}$, we obtain the $\mathbb{C}$-algebra $P_{k+\frac{1}{2}}(\xi, G)$.

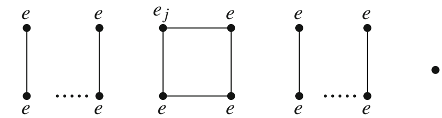
Similarly, the dimension of $\widehat{P}_{k+\frac{1}{2}}(x, G)$ is the number of $(G, k+\frac{1}{2})$-diagrams, so that if G is finite, $dim\ \widehat{P}_{k+\frac{1}{2}}(x, G) = |G|^{2k}B(2k+1)$. Define elements

$I^{(g_1,g_2,\ldots,g_k)} =$ 

$I_{(g_{k+1},g_{k+2},\ldots,g_{2k})} =$ 

$E_j =$ 

$\sigma_i =$ 

$\beta_j =$ 

where $g_l \in G$, $(1 \le l \le 2k)$, $(2 \le i \le k)$ and $(1 \le j \le k)$. Note that the last class in β_k is the triangle containing the last three vertices. We see that $\widehat{P}_{k+\frac{1}{2}}(x, G)$ is generated by the above elements and $P_{k+\frac{1}{2}}(x, G)$ is generated by the above elements except $I_{(g_{k+1},g_{k+2},\ldots,g_{2k})}$.

Theorem 3.2.1 *$P_{k+\frac{1}{2}}(x, G)$ is a subalgebra of $\widehat{P}_{k+\frac{1}{2}}(x, G)$.*

Proof In $\widehat{P}_{k+\frac{1}{2}}(x, G)$, for each $(G, k+\frac{1}{2})$-diagram (d_N, f) such that $f(1) = e$, we define the sum

$$\widehat{(d_N, f)} = \sum_{\overline{g} \in \overline{G}} (d_N, \overline{g} f).$$

In other words this sum is over all distinct $(G, k+\frac{1}{2})$-diagrams in $\widehat{P}_{k+\frac{1}{2}}(x, G)$, which are related to the $(G, k+\frac{1}{2})$-diagram (d_N, f) with respect to the above equivalence relation $\sim$ on $(G, k+\frac{1}{2})$-diagrams. So, we say that this sum is the class sum of (d_N, f) under $\sim$ in $\widehat{P}_{k+\frac{1}{2}}(x, G)$. Since any two class sums are the disjoint sums of $(G, k+\frac{1}{2})$-diagrams in $\widehat{P}_{k+\frac{1}{2}}(x, G)$ the set of all class sums is a linearly independent set in $\widehat{P}_{k+\frac{1}{2}}(x, G)$. We are going to prove that $(d_N, f) \longrightarrow \widehat{(d_N, f)}$ is an algebra isomorphism from $P_{k+\frac{1}{2}}(x, G)$ in to $\widehat{P}_{k+\frac{1}{2}}(x, G)$.

Now we have $(d_N, f)(d'_{N'}, f') = 0$ in $P_{k+\frac{1}{2}}(x, G)$

$\Leftrightarrow f_2' \neq \overline{g'} f_1, \forall \overline{g'} \in \overline{G}$
$\Leftrightarrow (d_N, \overline{g} f) * (d'_{N'}, \overline{g'} f') = 0$ in $\widehat{P}_{k+\frac{1}{2}}(x, G), \forall \overline{g}, \overline{g'} \in \overline{G}$
$\Leftrightarrow (\sum_{\overline{g} \in \overline{G}} (d, \overline{g} f)) * (\sum_{\overline{g'} \in \overline{G}} (d'_{N'}, \overline{g'} f')) = 0$ in $\widehat{P}_{k+\frac{1}{2}}(x, G)$
$\Leftrightarrow \widehat{(d_N, f)} * \widehat{(d'_{N'}, f')} = 0$ in $\widehat{P}_{k+\frac{1}{2}}(x, G)$.

Hence $(d_N, f)(d'_{N'}, f') = 0$ in $P_{k+\frac{1}{2}}(x, G) \Leftrightarrow \widehat{(d_N, f)} * \widehat{(d'_{N'}, f')} = 0$ in $\widehat{P}_{k+\frac{1}{2}}(x, G)$.

Suppose $(d_N, f)(d'_{N'}, f') \neq 0$. Let

$$(d_N, f)(d'_{N'}, f') = x^{\lambda}(d_N d'_{N'}, f''), \tag{3.10}$$

where λ is the number of middle components in $d_N d'_{N'}$. Then

$$\begin{aligned} \widehat{(d_N, f)} * \widehat{(d'_{N'}, f')} &= \left(\sum_{\overline{g} \in \overline{G}} (d_N, \overline{g} f) \right) * \left(\sum_{\overline{g'} \in \overline{G}} (d'_{N'}, \overline{g'} f') \right) \\ &= \sum_{\overline{g}, \overline{g'} \in \overline{G}} (d_N, \overline{g} f) * (d'_{N'}, \overline{g'} f'). \end{aligned} \tag{3.11}$$

Note that some product $(d_N, \overline{g} f) * (d'_{N'}, \overline{g'} f')$ may be 0 in the above sum (3.11). If a product $(d_N, \overline{g} f) * (d'_{N'}, \overline{g'} f')$ is non zero then the underlying $(G, k + \frac{1}{2})$-diagram must be $\sim$-equivalent to $(d_N d'_{N'}, f'')$ for some $f'' \in G^{2k}$ (using (3.10)). Also any $(G, k + \frac{1}{2})$- diagram which are $\sim$-equivalent to $(d_N d'_{N'}, f'')$ is of the form $(d_N, \overline{g} f) * (d'_{N'}, \overline{g'} f')$ for some $\overline{g}, \overline{g'} \in \overline{G}$. Observe that if $(d_N, \overline{g} f) * (d'_{N'}, \overline{g'} f') \neq 0$ then $\overline{g} f_2 = \overline{g'} f_1'$. And hence $(d_N, \overline{g} f) * (d'_{N'}, \overline{g'} f') = x^{\lambda}(d_N d'_{N'}, \overline{h} f'')$ for some $\overline{h} \in \overline{G}$. Hence (3.11) can be written as

$$\sum_{\overline{h} \in \overline{G}} x^{\lambda}(d_N d'_{N'}, \overline{h} f'') = (d_N, \widehat{f)(d'_{N'}}, f').$$

Thus $\widehat{(d_N, f)} * \widehat{(d'_{N'}, f')} = (d_N, \widehat{f)(d'_{N'}}, f')$.

Hence $Span_{\mathbb{C}(x)}\{\widehat{(d_N, f)} \mid (d_N, f)$ is a $(G, k + \frac{1}{2})$-diagram such that $f(1) = e\}$ is a subalgebra and the map $(d_N, f) \longrightarrow \widehat{(d_N, f)}$ is an algebra isomorphism from $P_{k+\frac{1}{2}}(x, G)$ in to $\widehat{P}_{k+\frac{1}{2}}(x, G)$. □

3.3 Schur–Weyl duality

We have the diagonal action of $G \times S_{n-1}$ on $W^{\otimes k}$, where W is the permutation representation of S_n. Also, we have an action of $\widehat{P}_{k+\frac{1}{2}}(n, G)$ on $W^{\otimes k}$, defined as follows: Define a map $\widehat{\phi} : \widehat{P}_{k+\frac{1}{2}}(n, G) \longrightarrow End(W^{\otimes k})$ by defining it on a $(G, k+\frac{1}{2})$- diagram (d_N, f), where $f = (g_1, g_2, \ldots, g_k, g_{k+1}, g_{k+2}, \ldots, g_{2k})$ is the label sequence of d_N as follows:

$$\widehat{\phi}(d_N, f) = \sum_{\substack{p\sim q \ in \ d \Rightarrow i_p = i_q \\ p\in N \Rightarrow i_p = n}} E^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}.$$

Then we have an action of $\widehat{P}_{k+\frac{1}{2}}(n, G)$ on $W^{\otimes k}$ defined by

$$d(v_I) = \widehat{\phi}(d)(v_I), \forall I \in \mathbb{S}^k.$$

When G is a group with one element, this action restricts to the action of the rook partition algebra on tensors.

The multiplication of the matrices $\widehat{L}^I_J$ in the basis of $End_{S_{n-1}}(W^{\otimes k})$ has a nice form as follows:

Lemma 3.3.1

(a) $\left(L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}\right)\left(L^{(j_1,h_1),(j_2,h_2),\ldots,(j_k,h_k)}_{(j_{k+1},h_{k+1}),(j_{k+2},h_{k+2}),\ldots,(j_{2k},h_{2k})}\right) = 0$
$\Leftrightarrow g(g_1, g_2, \ldots, g_k) \neq (h_{k+1}, h_{k+2}, \ldots, h_{2k})$ *for someg* $\in G$.

(b) $\left(\widetilde{L}^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}\right)\left(\widetilde{L}^{(j_1,h_1),(j_2,h_2),\ldots,(j_k,h_k)}_{(j_{k+1},h_{k+1}),(j_{k+2},h_{k+2}),\ldots,(j_{2k},h_{2k})}\right) = 0$
$\Leftrightarrow (g_1, g_2, \ldots, g_k) \neq (h_{k+1}, h_{k+2}, \ldots, h_{2k})$.

Proof a)

$$\begin{aligned}
&\left(L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}\right)\left(L^{(j_1,h_1),(j_2,h_2),\ldots,(j_k,h_k)}_{(j_{k+1},h_{k+1}),(j_{k+2},h_{k+2}),\ldots,(j_{2k},h_{2k})}\right) \\
&= \left(\sum_{\substack{i_p=i_q\Rightarrow i'_p=i'_q \\ i_p=n\Rightarrow i'_p=n, \quad g\in G}} E^{(i'_1,gg_1),(i'_2,gg_2),\ldots,(i'_k,gg_k)}_{(i'_{k+1},gg_{k+1}),\ldots,(i'_{2k},gg_{2k})}\right) \\
&\quad \times \left(\sum_{\substack{j_p=j_q\Rightarrow j'_p=j'_q \\ j_p=n\Rightarrow j'_p=n, \quad h\in H}} E^{(j'_1,hh_1),(j'_2,hh_2),\ldots,(j'_k,hh_k)}_{(j'_{k+1},hh_{k+1}),\ldots,(j_{2k},hh_{2k})}\right) \\
&= \sum E^{(i'_1,gg_1),(i'_2,gg_2),\ldots,(i'_k,gg_k)}_{(i'_{k+1},gg_{k+1}),(i'_{k+2},gg_{k+2}),\ldots,(i'_{2k},gg_{2k})} E^{(j'_1,hh_1),(j'_2,hh_2),\ldots,(j'_k,hh_k)}_{(j'_{k+1},hh_{k+1}),(j'_{k+2},hh_{k+2}),\ldots,(j'_{2k},hh_{2k})} \\
&= \sum \delta^{(i'_1,gg_1),(i'_2,gg_2),\ldots,(i'_k,gg_k)}_{(j'_{k+1},hh_{k+1}),(j'_{k+2},hh_{k+2}),\ldots,(j'_{2k},hh_{2k})} E^{(j'_1,hh_1),(j'_2,hh_2),\ldots,(j'_k,hh_k)}_{(i'_{k+1},gg_{k+1}),(i'_{k+2},gg_{k+2}),\ldots,(i'_{2k},gg_{2k})}, \\
&\qquad (\text{since} E_{pq}E_{rs} = \delta_{qr}E_{ps}, \text{ where } \delta_{qr} \text{ is the Kronecker delta}) \\
&= 0 \quad \text{if and only if} \quad g(g_1, g_2, \ldots, g_k) \neq (h_{k+1}, h_{k+2}, \ldots, h_{2k}) \text{ for some } g \in G.
\end{aligned}$$

(b)

$$\left(\widetilde{L}^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}\right)\left(\widetilde{L}^{(j_1,h_1),(j_2,h_2),\ldots,(j_k,h_k)}_{(j_{k+1},h_{k+1}),(j_{k+2},h_{k+2}),\ldots,(j_{2k},h_{2k})}\right)$$

$$= \left(\sum_{\substack{i_p=i_q \Rightarrow i'_p=i'_q \\ i_p=n \Rightarrow i'_p=n}} E^{(i'_1,g_1),(i'_2,g_2),\dots,(i'_k,g_k)}_{(i'_{k+1},g_{k+1}),\dots,(i'_{2k},g_{2k})} \right) \left(\sum_{\substack{j_p=j_q \Rightarrow j'_p=j'_q \\ j_p=n \Rightarrow j'_p=n}} E^{(j'_1,h_1),(j'_2,h_2),\dots,(j'_k,h_k)}_{(j'_{k+1},h_{k+1}),\dots,(j_{2k},h_{2k})} \right)$$
$$= \sum E^{(i'_1,g_1),(i'_2,g_2),\dots,(i'_k,g_k)}_{(i'_{k+1},g_{k+1}),(i'_{k+2},g_{k+2}),\dots,(i'_{2k},g_{2k})} E^{(j'_1,h_1),(j'_2,h_2),\dots,(j'_k,h_k)}_{(j'_{k+1},h_{k+1}),(j'_{k+2},h_{k+2}),\dots,(j'_{2k},h_{2k})}$$
$$= \sum \delta^{(i'_1,g_1),(i'_2,g_2),\dots,(i'_k,g_k)}_{(j'_{k+1},h_{k+1}),(j'_{k+2},h_{k+2}),\dots,(j'_{2k},h_{2k})} E^{(j'_1,h_1),(j'_2,h_2),\dots,(j'_k,h_k)}_{(i'_{k+1},g_{k+1}),(i'_{k+2},g_{k+2}),\dots,(i'_{2k},g_{2k})}$$
$$= 0 \text{ if and only if} (g_1, g_2, \dots, g_k) \neq (h_{k+1}, h_{k+2}, \dots, h_{2k}).$$

□

Lemma 3.3.2 (a) *For each $g \in G$,*

$$\left(L^{(i_1,g_1),(i_2,g_2),\dots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\dots,(i_{2k},g_{2k})}\right) \left(L^{(j_1,h_1),(j_2,h_2),\dots,(j_k,h_k)}_{(j_{k+1},gg_1),(j_{k+2},gg_2),\dots,(j_{2k},gg_k)}\right)$$
$$= (n)^{\lambda} \; L^{(s_1,h_1),(s_2,h_2),\dots,(s_k,h_k)}_{(s_{k+1},gg_{k+1}),(s_{k+2},gg_{k+2}),\dots,(s_{2k},gg_{2k})}, \tag{3.12}$$

and (b) $$\left(\widetilde{L}^{(i_1,g_1),(i_2,g_2),\dots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\dots,(i_{2k},g_{2k})}\right) \left(\widetilde{L}^{(j_1,h_1),(j_2,h_2),\dots,(j_k,h_k)}_{(j_{k+1},g_1),(j_{k+2},g_2),\dots,(j_{2k},g_k)}\right)$$
$$= (n)^{\lambda} \; \widetilde{L}^{(s_1,h_1),(s_2,h_2),\dots,(s_k,h_k)}_{(s_{k+1},g_{k+1}),(s_{k+2},g_{k+2}),\dots,(s_{2k},g_{2k})},$$

where λ is the number of middle components in the product $d_N d'_{N'}$ (where d_N and $d'_{N'}$ are defined in Lemma 3.2.2*) and $(1 \leq s_1, s_2, \dots, s_{2k} \leq n)$ such that $p \sim q$ in $d_N d'_{N'} \Leftrightarrow s_p = s_q$.*

Proof (a) For each $g \in G$

$$\left(L^{(i_1,g_1),(i_2,g_2),\dots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\dots,(i_{2k},g_{2k})}\right) \left(L^{(j_1,h_1),(j_2,h_2),\dots,(j_k,h_k)}_{(j_{k+1},gg_1),(j_{k+2},gg_2),\dots,(j_{2k},gg_k)}\right)$$
$$= \left(\sum_{\substack{i_p=i_q \Rightarrow i'_p=i'_q \\ i_p=n \Rightarrow i'_p=n, \quad g' \in G}} E^{(i'_1,g'g_1),(i'_2,g'g_2),\dots,(i'_k,g'g_k)}_{(i'_{k+1},g'g_{k+1}),\dots,(i'_{2k},g'g_{2k})} \right)$$
$$\times \left(\sum_{\substack{j_p=j_q \Rightarrow j'_p=j'_q \\ j_p=n \Rightarrow j'_p=n, \quad h' \in G}} E^{(j'_1,h'h_1),(j'_2,h'h_2),\dots,(j'_k,h'h_k)}_{(j'_{k+1},h'g_1),\dots,(j'_{2k},h'g_{2k})} \right)$$
$$= \sum E^{(i'_1,g'g_1),(i'_2,g'g_2),\dots,(i'_k,g'g_k)}_{(i'_{k+1},g'g_{k+1}),\dots,(i'_{2k},g'g_{2k})} \quad \sum E^{(j'_1,h'h_1),(j'_2,h'h_2),\dots,(j'_k,h'h_k)}_{(j'_{k+1},h'g_1),\dots,(j'_{2k},h'g_{2k})}$$

$$= \sum \delta^{(j'_{k+1},hg'g_1),(j'_{k+2},hg'g_2),\ldots,(j'_{2k},hg'g_{2k})}_{(i'_1,gg_1),(i'_2,gg_2),\ldots,(i'_k,gg_k)} E^{(j'_1,hh_1),(j'_2,hh_2),\ldots,(j'_k,hh_k)}_{(i'_{k+1},gg_{k+1}),(i'_{k+2},gg_{k+2}),\ldots,(i'_{2k},gg_{2k})}.$$

$$= \sum \delta^{(j'_{k+1},j'_{k+2},\ldots,j'_{2k})}_{(i'_1,i'_2,\ldots,i'_k)} E^{(j'_1,hh_1),(j'_2,hh_2),\ldots,(j'_k,hh_k)}_{(i'_{k+1},hg'g_{k+1}),(i'_{k+2},hg'g_{k+2}),\ldots,(i'_{2k},hg'g_{2k})}. \tag{3.13}$$

The number of times $E^{(j'_1,hh_1),(j'_2,hh_2),\ldots,(j'_k,hh_k)}_{(i'_{k+1},hg'g_{k+1}),(i'_{k+2},hg'g_{k+2}),\ldots,(i'_{2k},hg'g_{2k})}$ appears in the above sum is equal to the number of pairs of sequences

$$\begin{cases} j'_{k+1}, j'_{k+2}, \ldots, j'_{2k}, & j'_1, \quad j'_2, \ldots, \qquad j'_k \\ i'_1, \quad i'_2, \ldots \quad, i'_k, & i'_{k+1}, i'_{k+2}, \ldots, i'_{2k} \end{cases}. \tag{3.14}$$

such that

(i) $i_p = i_q \Rightarrow i'_p = i'_q$ and $i_p = n \Rightarrow i'_p = n$

(ii) $j_p = j_q \Rightarrow j'_p = j'_q$ and $j_p = n \Rightarrow j'_p = n$

(iii) $i'_1, i'_2, \ldots, i'_k$ and $j'_{k+1}, j'_{k+2}, \ldots, j'_{2k}$ are fixed.

(iv) $i'_{k+1} = j'_1, i'_{k+2} = j'_2, \ldots, i'_{2k} = j'_k$

Suppose there is a middle component in the product $d_N d'_{N'}$ then we can give n values for the corresponding component in Eq. (3.14) to get different sequence. Hence (3.13) can be written as in (3.14). This comes from a sequence of purely combinatorial arguments as in the case of the partition algebra. Proof of (b) is similar to the proof of (a). □

Theorem 3.3.3 (a) *The map* $\widehat{\phi} : \widehat{P}_{k+\frac{1}{2}}(n, G) \longrightarrow End(W^{\otimes k})$ *is an algebra homomorphism onto on* $End_{S_{n-1}} W^{\otimes k}$.
(b) *The restricted map* $\phi : P_{k+\frac{1}{2}}(n, G) \longrightarrow End(W^{\otimes k})$ *is onto on* $End_{G\times S_{n-1}} W^{\otimes k}$.

Proof (a) It is enough to prove $\widehat{\phi}\{(d_N, f) * (d'_{N'}, f')\} = \widehat{\phi}(d_N, f)\widehat{\phi}(d_{N'}, f')$, where (d_N, f), $(d_{N'}, f')$ are $(G, k+\frac{1}{2})$-diagrams. So the homomorphism is an immediate consequence of Lemma 3.3.1 and Lemma 3.3.2. Note that each $\widetilde{L}^I_J$ has pre image such that $\widehat{\phi}(d(i_1, i_2, \ldots, i_{2k})_N, f) = \widetilde{L}^{(i_1,f(1),(i_2,f(2),\ldots,(i_k,f(k)}_{(i_{k+1},f(k+1),(i_{k+2},f(k+2),\ldots,(i_{2k},f(2k)}$, where $N = \{p \in [2k] \mid i_p = n\}$.
(b) From Lemma 3.2.1, We have

$$\widehat{\phi}(\widehat{d_N, f}) = \widehat{\phi}\left(\sum_{\overline{g}\in\overline{G}} (d_N, \overline{g} f)\right)$$
$$= \sum_{\overline{g}\in\overline{G}} \widehat{\phi}(d_N, \overline{g} f)$$

$$= \sum_{g \in G} \sum_{\substack{x \sim y \ in \ d \Rightarrow j_x = j_y \\ x \in N \Rightarrow i_x = n}} E^{(j_1, gf(1)), (j_2, gf(2)), \ldots, (j_k, gf(k))}_{(j_{k+1}, gf(k+1)), (j_{k+2}, gf(k+2)), \ldots, (j_{2k}, gf(2k))}$$

$$= \sum_{\substack{x \sim y \ in \ d \Rightarrow j_x = j_y \\ g \in G, \ x \in N \Rightarrow i_x = n}} E^{(j_1, gf(1)), (j_2, gf(2)), \ldots, (j_k, gf(k))}_{(j_{k+1}, gf(k+1)), (j_{k+2}, gf(k+2)), \ldots, (j_{2k}, gf(2k))}.$$

Thus the matrices $\widehat{\phi}(\widehat{d_N, f}) \in End_{G \times s_{n-1}}$ (Using (3.4)). Note that each L^I_J has pre image such that $\widehat{\phi}(d(i_1, i_2, \widehat{\ldots}, i_{2k})_N, f) = L^{(i_1, f(1), (i_2, f(2), \ldots, (i_k, f(k)}_{(i_{k+1}, f(k+1), (i_{k+2}, f(k+2), \ldots, (i_{2k}, f(2k)}$, where $N = \{p \in [2k] \mid i_p = n\}$. □

Corollary 3.3.4 *S_{n-1} and $\widehat{P}_{k+\frac{1}{2}}(n, G)$ generate full centralizers of each other in $End(W^{\otimes k})$. In particular, for $n-1 \geq 2k$, we have* (a) $\widehat{P}_{k+\frac{1}{2}}(n, G) \cong End_{S_{n-1}}(W^{\otimes k})$, (b) $\mathbb{C}(S_{n-1})$ *generates* $End_{\widehat{P}_{k+\frac{1}{2}}}(W^{\otimes k})$.

Proof The proof is the immediate consequence of Theorem 3.3.3. Since $n - 1 \geq 2k$, $dim\ \widehat{P}_{k+\frac{1}{2}}(n, G) = dim\ End_{S_{n-1}}(W \otimes k)$. In the proof of Theorem 3.3.3, we have $\widehat{\phi}(\widehat{P}_{k+\frac{1}{2}}(n, G)) \subseteq End_{S_{n-1}}(W^{\otimes k})$. As (d_N, f) ranges over all diagrams, all $\widehat{L}_d$ are obtained. Thus the representation $\widehat{\phi}$ takes a basis of $\widehat{P}_{k+\frac{1}{2}}(n, G)$ to a basis of $End_{S_{n-1}}(W^{\otimes k})$, so $\widehat{P}_{k+\frac{1}{2}}(n, G) \cong End_{S_{n-1}}(W^{\otimes k})$. Proof of (b): This follows from (a) and the double centralizer Theorem. □

Corollary 3.3.5 *$G \times S_{n-1}$ and $P_{k+\frac{1}{2}}(n, G)$ generate full centralizers of each other in $End(W^{\otimes k})$. In particular, for $n - 1 \geq 2k$, we have* (a) $P_{k+\frac{1}{2}}(n, G) \cong End_{G \times S_{n-1}}(W^{\otimes k})$, (b) $\mathbb{C}(G \times S_{n-1})$ *generates* $End_{P_{k+\frac{1}{2}}}(W^{\otimes k})$.

Proof The proof is similar to the Corollary 3.3.3. □

As centralizers of the semisimple group algebras $\mathbb{C}(S_{n-1})$ and $\mathbb{C}(G \times S_{n-1})$ respectively, the $\mathbb{C}$-algebras $\widehat{P}_{k+\frac{1}{2}}(n, G)$ and $P_{k+\frac{1}{2}}(n, G)$ are semisimple for $n - 1 \geq 2k$.

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